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Identification of multiple cracks in a beam using vibration amplitudes

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ABSTRACT

A simple method to identify multiple cracks in a beam using the vibration amplitudes is presented. The cracks are modeled as massless rotational springs and the forward problem is solved using the finite element method. The inverse problem is solved iteratively for the crack locations and sizes using the Newton–Raphson method and the singular value decomposition method. A two-dimensional finite element model is built to simulate the experimental results and to provide vibration amplitude measurements. The difficulty of identifying multiple cracks without *a priori* knowledge of the number of cracks is overcome by comparing the residual sum of squares of each solution with assumed number of cracks. The crack locations are estimated accurately and the accuracy of the crack size estimations are to be enhanced greatly by an improved torsional stiffness model.

1. Introduction

Due to its practical importance, the crack identification problem in structures has been extensively investigated and many methods were proposed. Despite of extensive research on the crack identification based on the changes of natural frequencies [1–14], little work has been done using the changes of vibration amplitudes.

The frequency contour plot method [1–7] had been one of the most favored tools to identify a single crack using the lowest three natural frequencies, which was further developed to deal with multiple cracks [8]. Liang et al. [1] proposed the frequency contour method in which the crack was modeled as a massless rotational spring. The solution of a characteristic equation for a given natural frequency and a crack location gave the local stiffness. Because only one value of stiffness was permissible at a given crack location, the intersections of the various values of stiffness at various natural frequencies along the axial direction provided the location of the crack. The scheme was applied to the crack detection in stepped beams [2] and truncated wedged beams [3,4]. Nandwana and Maiti [5] modeled the vibration of a beam in the presence of an inclined edge or internal normal crack to detect the location of the crack. Lele and Maiti [6] extended the frequency contour plot method to the crack identification in beams based on the Timoshenko beam theory. Nikolakopoulos et al. [7] identified the crack location and the crack depth of frame structures by determining the superposed contour intersections. Hu and Liang [8] proposed two damage modeling techniques, one involving the use of massless spring to represent the discrete cracks and the other one employing the continuum concept. The continuum model was used first to identify the discretizing element of a structure that contained cracks, and then the spring damage model was used to quantify the location and size

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of the crack in each damaged element. Patil and Maiti [9] adopted the approach of Hu and Liang [8] and applied the transfer matrix method to the identification of multiple cracks.

Narkis [10] calculated the natural frequencies of a cracked simply supported beam by an approximate analytical solution and proposed that the only information required for the crack identification was the variation of the first two natural frequencies due to the crack. Morassi [11] derived an explicit expression of the frequency sensitivity to the crack if the crack was very small, and proposed that the measurement of the frequency of cracked beam under simply supported boundary conditions determined uniquely the stiffness of the rotational spring and the position. Dado [12] proposed a two-table method for the detection of the crack. The first table was entered with the two natural frequencies to obtain the crack location and the second table was entered with the first natural frequency and the crack location to obtain the crack depth. Shifrin and Ruotolo [13] proposed that $k+2$ equations were sufficient to form the system determinant for a beam with k cracks. Lee [14] applied the Newton–Raphson method to identify k cracks in a beam where $2k$ natural frequencies of the cracked beam were required. The elements of the Jacobian matrix were the sensitivities of the natural frequencies with respect to the crack parameters.

Owolabi et al. [15] used the changes in natural frequencies and frequency response function amplitudes as a function of crack depths and locations in the crack detection. They noticed that the mode shape underwent a noticeable change close to the crack location area as the crack grew, and estimated the crack depth and location based on the observed changes in the natural frequencies and mode shapes. Rizos et al. [16] estimated the crack size and location from the measured amplitudes of the structure vibrating at one of its natural modes and the analytical solution of the dynamic response. Dilena and Morassi [17] proposed a method to detect a single crack when damage-induced shifts in the nodes of the mode shapes of a beam were known.

The objective of the present study is to present a simple method of identifying multiple cracks in a beam using the changes of the forced vibration amplitudes. The Newton–Raphson method and the singular value decomposition method are used for the estimation of crack parameters.

2. Forward problem

The geometry of a beam with multiple cracks is given in Fig. 1. The cracks are represented by massless rotational springs. Parameters $\alpha_i = a_i/h$ and $\beta_i = s_i/L\,(i = 1, 2, \ldots)$ denote the normalized crack size and the normalized crack location. h and L are the thickness and the length of the beam. The finite element equation of a beam element of length ΔL based on the Euler–Bernoulli theory is given as

$$[\mathbf{M}]^e\{\ddot{W}\}^e + [\mathbf{K}]^e\{W\}^e = \{F\}^e \tag{1}$$

where matrices $[\mathbf{M}]^e$ and $[\mathbf{K}]^e$ are the element mass and stiffness matrices defined as

$$[\mathbf{M}]^e = \frac{\rho A \Delta L}{420}\begin{bmatrix} 156 & 22\Delta L & 54 & -13\Delta L \\ & 4(\Delta L)^2 & 13\Delta L & -3(\Delta L)^2 \\ & & 156 & -22\Delta L \\ SYM & & & 4(\Delta L)^2 \end{bmatrix} \tag{2}$$

and

$$[\mathbf{K}]^e = \frac{EI}{(\Delta L)^3}\begin{bmatrix} 12 & 6\Delta L & -12 & 6\Delta L \\ & 4(\Delta L)^2 & -6\Delta L & 2(\Delta L)^2 \\ & & 12 & -6\Delta L \\ SYM & & & 4(\Delta L)^2 \end{bmatrix} \tag{3}$$

$\{W\}^e = \{w_i \;\; \theta_i \;\; w_{i+1} \;\; \theta_{i+1}\}^{\mathrm{T}}$ and $\{F\}^e$ are the element generalized displacement vector and element generalized load vector. E, ρ, A and I are Young's modulus, the density, the cross-sectional area and the second moment of area, respectively.

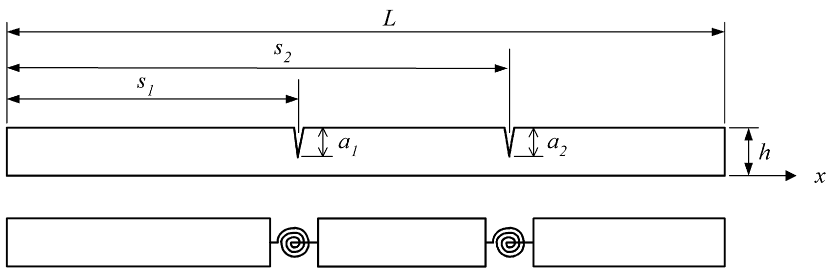

Fig. 1. Beam with cracks and massless rotational spring model.

When the massless rotational spring connects node j and node $j+1$, the deflections of node j and node $j+1$ are the same, i.e., $w_j = w_{j+1}$. It is also required that the rotations θ_j and θ_{j+1} are coupled through the cracked stiffness matrix

$$[\mathbf{K}]_c = \begin{bmatrix} \kappa & -\kappa \\ -\kappa & \kappa \end{bmatrix} \tag{4}$$

κ is the torsional stiffness at the open crack which was proposed by Ostachowicz and Krawkczuk [18] as

$$\kappa_1 = \frac{bh^2E}{72\pi\alpha^2 f_1(\alpha)} \tag{5a}$$

$$f_1(\alpha) = 0.6384 - 1.035\alpha + 3.7201\alpha^2 - 5.1773\alpha^3 + 7.553\alpha^4 - 7.332\alpha^5 + 2.4909\alpha^6 \tag{5b}$$

and it was employed in the studies of Nandwana and Maiti [2], Chaudhari and Maiti [3], and Lele and Maiti [6] and Patil and Maiti [9]. b stands for the beam width. On the other hand, Dimarogonas [19] and Dimarogonas and Paipetis [20] proposed κ as

$$\kappa_2 = \frac{EI}{5.346hf_2(\alpha)} \tag{6a}$$

$$f_2(\alpha) = 1.8624\alpha^2 - 3.95\alpha^3 + 16.375\alpha^4 - 37.226\alpha^5 + 76.81\alpha^6 - 126.9\alpha^7 + 172\alpha^8 - 143.97\alpha^9 + 66.56\alpha^{10} \tag{6b}$$

which was adopted in the works of Hu and Liang [8], Dado [12], Shifrin and Ruotolo [13], Rizos et al. [16] and Ruotolo and Surace [21].

Matrices $[\mathbf{M}]^e$, $[\mathbf{K}]^e$ and $[\mathbf{K}]_c$ are assembled to form the global mass and stiffness matrices $[\mathbf{M}]$ and $[\mathbf{K}]$, and the equations of motion becomes

$$[\mathbf{M}]\{\ddot{W}\} + [\mathbf{K}]\{W\} = \{F(t)\} \tag{7}$$

When the beam is exited by a sinusoidal force at frequency ω_i rad/s

$$\{F(t)\} = \{F^*\} \sin \omega_i t \tag{8}$$

the global generalized displacement vector $\{W\}$ is also in a sinusoidal motion

$$\{W\} = \{W^*\} \sin \omega_i t \tag{9}$$

in the absence of damping, and the equations of motion of Eq. (7) becomes

$$([\mathbf{K}] - \omega_i^2[\mathbf{M}])\{W^*\} = \{F^*\} \tag{10}$$

Let us assume one of the simplest setups of forced vibration. A shaker is installed at the free end ($x = L$) of a cantilever beam and excites the beam at m different frequencies $(\omega_1, \omega_2, \ldots, \omega_m)$ and the vertical deflections are observed at n evenly spaced locations as shown in Fig. 2. Also define w_{ij} to be the deflection at location j divided by the deflection of the free end when the beam is excited at frequency ω_i rad/s. A program is written in Matlab to compute the deflection w_{ij} when the beam is excited at frequency ω_i rad/s. It is worthwhile to note that the excitation frequency ω_i does not have to be one of the natural frequencies of the beam.

Ruotolo and Surace [21] conducted experimental tests on cantilever beams with and without cracks and measured the natural frequencies as listed in Table 1. The cantilevers were made of C30 steel and their dimensions were $0.02 \times 0.02 \times 0.8\,\text{m}^3$. The cracked beam had double cracks and the crack parameters were $\alpha_1 = 0.2$, $\alpha_2 = 0.3$, $\beta_1 = 0.3182$ and $\beta_2 = 0.6812$. The cracks were obtained by wire erosion with a 0.1 mm diameter wire to produced notches 0.13 mm wide.

In general Young's modulus of steel is considered to range from 180 to 210 GPa. Alternatively, Young's modulus can be found from the measured natural frequencies of the undamaged cantilever beam and the Euler–Bernoulli beam theory

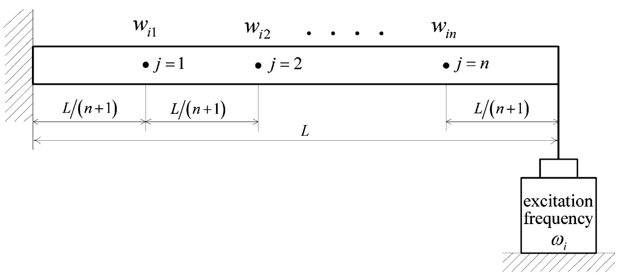

Fig. 2. Schematic diagram of deflection w_{ij} at excitation frequency ω_i.

Table 1
Natural frequencies of undamaged and cracked beams.

			f_1 (Hz)	f_2 (Hz)	f_3 (Hz)
Experimental measurements [21]		Undamaged	24.175	152.103	424.455
		Cracked	24.044	149.268	409.287
Present study ($E = 181$ GPa)	2D model (ANSYS)	Cracked	24.108	149.09	408.73
	$\kappa = \kappa_1$	Cracked	24.105	149.54	412.04
	$\kappa = \kappa_2$	Cracked	24.145	150.12	415.02
	$\kappa = 0.83\kappa_1$	Cracked	24.067	149.01	409.33
	$\kappa = 0.65\kappa_2$	Cracked	24.067	149.03	409.39

C30 steel, $b = 0.02$ m, $h = 0.02$ m, $L = 0.8$ m, double cracks, $\alpha_1 = 0.2$, $\alpha_2 = 0.3$, $\beta_1 = 0.3182$, and $\beta_2 = 0.6812$.

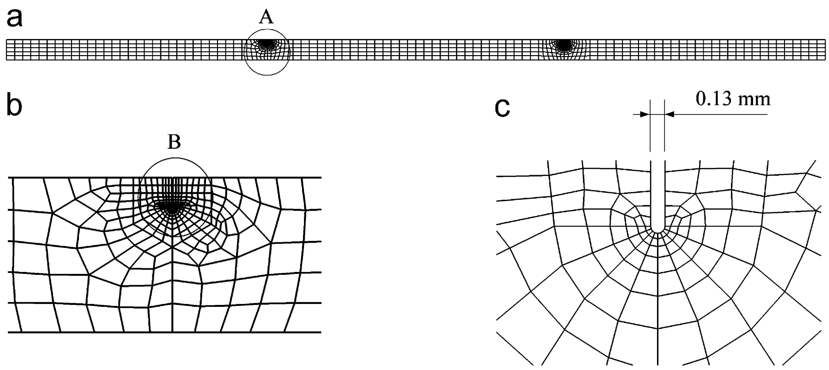

Fig. 3. Two-dimensional finite element model for ANSYS analysis (1135 eight-node elements, 3732 nodes, crack width of 0.13 mm, $L = 0.8$ m, $h = 0.02$ m, $\alpha_1 = 0.2$, $\alpha_2 = 0.3$, $\beta_1 = 0.3182$, and $\beta_2 = 0.6812$).

relationships

$$f_1 = \frac{1.875^2}{2\pi}\sqrt{\frac{EI}{\rho AL^4}}, \quad f_2 = \frac{4.694^2}{2\pi}\sqrt{\frac{EI}{\rho AL^4}}, \quad f_3 = \frac{7.855^2}{2\pi}\sqrt{\frac{EI}{\rho AL^4}} \tag{11a,b,c}$$

The averaged Young's modulus computed from the measured natural frequencies of the undamaged cantilever of Table 1 and Eqs. (11a–c) is 181 GPa, and $E = 181$ GPa is used through out this study.

To simulate the experimental results of Ruotolo and Surace [21] a finite element analysis is carried out using the commercial program ANSYS. A two-dimensional mesh composed of 1135 eight-node elements and 3732 nodes is built as shown in Fig. 3, and the material properties $E = 181$ GPa, $\rho = 7860\,\text{kg/m}^3$ and Poisson's ratio $\nu = 0.29$ are input. The solution from ANSYS is given in Table 1, which are in good agreement with the measured results of Ruotolo and Surace [21]. It indicates that the two-dimensional ANSYS model accurately simulates the dynamic characteristics of the experimental results.

The torsional stiffness models of Eqs. (5) and (6) are compared in Fig. 4. It shows that the discrepancy between Eqs. (5) and (6) in the range of $(0.1 < \alpha < 0.5)$ is significant, which implies that the application of them without proper calibration may lead to an erroneous solution.

The natural frequency ω in rad/s of the Euler–Bernoulli beam with rotational spring model can be computed from the free vibration equation

$$[\mathbf{K}]\{W\} = \omega^2[\mathbf{M}]\{W\} \tag{12}$$

where matrices [**M**] and [**K**] are borrowed from Eq. (7). Another program is written in Matlab, and the natural frequencies of the cantilever beam with the rotational spring model $\kappa = \kappa_1$ (the torsional stiffness model of Ostachowicz and Krawkczuk [18]) and $\kappa = \kappa_2$ (the torsional stiffness model of Dimarogonas and Paipetis [20]) are computed, respectively, and the results are given in Table 1. It shows that the natural frequencies of the beam with either the rotational spring model $\kappa = \kappa_1$ or $\kappa = \kappa_2$ deviate considerably from the measured data and that both the torsional stiffness models need improvements. The proper selection of the torsional stiffness model is one of the most crucial factors in the crack detection in structures. Investigation of the sophisticated torsional stiffness model is beyond the scope of this study, and simple scaling down of κ_1 and κ_2 is considered. After several trials it is found that the torsional stiffness model $\kappa = 0.83\kappa_1$ or $\kappa = 0.65\kappa_2$ provides better results than $\kappa = \kappa_1$ or $\kappa = \kappa_2$ as shown in Table 1.

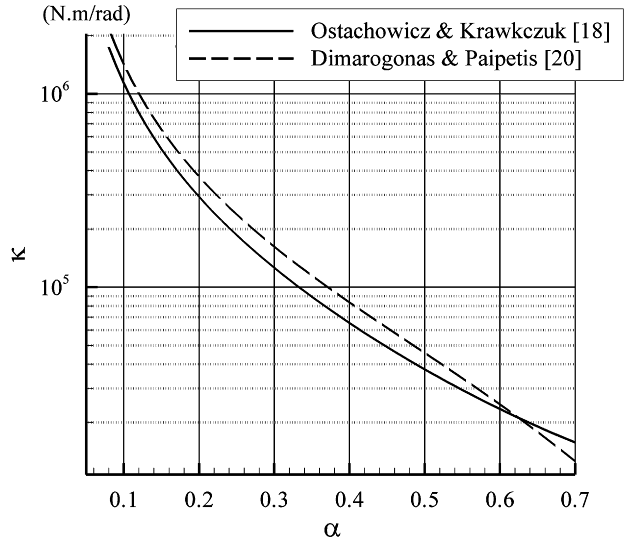

Fig. 4. Comparison of the torsional stiffness models of Ostachowicz and Krawkczuk [18] and Dimarogonas and Paipetis [20] ($E = 181$ GPa, $b = 0.02$ m, and $h = 0.02$ m).

Table 2
Vibration amplitudes w_{ij} of a cantilever beam by harmonic response analysis of ANSYS and two-dimensional mesh of Fig. 3.

Excitation frequency ω_i (rad/s)	w_{i1}	w_{i2}	w_{i3}
250 ($i = 1$)	0.1192	0.3921	0.7058
500 ($i = 2$)	0.3786	0.9873	1.2365
1000 ($i = 3$)	−0.3753	−0.6074	−0.0472
2000 ($i = 4$)	−2.3603	−1.2285	2.4105

$L = 0.8$ m, $d = 0.02$ m, $h = 0.02$ m, $E = 181$ GPa, $\rho = 7860\,\mathrm{kg/m^3}$, $\nu = 0.29$, double cracks, $\alpha_1 = 0.2$, $\alpha_2 = 0.3$, $\beta_1 = 0.3182$, $\beta_2 = 0.6812$, $m = 4$, and $n = 3$.

Because the solution by ANSYS in Table 1 is proved reliable, the deflections w_{ij}'s are obtained by the harmonic response analyses using ANSYS and the two-dimensional finite element model of Fig. 3 and the results are given in Table 2. The same material properties are used as in the computation of the natural frequencies. The number of excitation frequencies and the number of amplitude pickup points in the beam are $m = 4$ and $n = 3$. The excitation frequencies are $\omega_1 = 250$ rad/s, $\omega_2 = 500$ rad/s, $\omega_3 = 1000$ rad/s and $\omega_4 = 2000$ rad/s.

3. Inverse problem

It is assumed that mn vibration amplitude measurements $(w^o_{11}, w^o_{12}, \ldots, w^o_{mn})$ are available. For the identification of k cracks there are $2k$ unknown crack parameters: α_1, β_1, α_2, β_2,…,α_k and β_k ($\beta_1 < \beta_2 < \cdots < \beta_k$). The Newton–Raphson procedure is applied as follows:

(a) assume the initial values of $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_k, \beta_k$;
(b) locate the nodes that represent the cracks according to the new crack position parameters $\beta_1, \beta_2, \ldots, \beta_k$ and generate the finite element mesh of the beam;
(c) solve the forward problem for the vibration amplitudes $w_{11}, w_{12}, \ldots, w_{mn}$ with the given crack parameters $(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_k, \beta_k)$ and excitation frequencies $(\omega_1, \omega_2, \ldots, \omega_m)$, and evaluate the Jacobian matrix $[J]$:

$$[J] = \begin{bmatrix} \frac{\partial w_{11}}{\partial \alpha_1} & \frac{\partial w_{11}}{\partial \beta_1} & \frac{\partial w_{11}}{\partial \alpha_2} & \frac{\partial w_{11}}{\partial \beta_2} & \cdots & \frac{\partial w_{11}}{\partial \alpha_k} & \frac{\partial w_{11}}{\partial \beta_k} \\ \frac{\partial w_{12}}{\partial \alpha_1} & \frac{\partial w_{12}}{\partial \beta_1} & \frac{\partial w_{12}}{\partial \alpha_2} & \frac{\partial w_{12}}{\partial \beta_2} & \cdots & \frac{\partial w_{12}}{\partial \alpha_k} & \frac{\partial w_{12}}{\partial \beta_k} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{\partial w_{mn}}{\partial \alpha_1} & \frac{\partial w_{mn}}{\partial \beta_1} & \frac{\partial w_{mn}}{\partial \alpha_2} & \frac{\partial w_{mn}}{\partial \beta_2} & \cdots & \frac{\partial w_{mn}}{\partial \alpha_k} & \frac{\partial w_{mn}}{\partial \beta_k} \end{bmatrix} \tag{13}$$

and the residuals

$$\begin{cases} \Re_{11} = w_{11} - w_{11}^{o} \\ \Re_{12} = w_{12} - w_{12}^{o} \\ \vdots \\ \Re_{mn} = w_{mn} - w_{mn}^{o} \end{cases} \tag{14}$$

(d) solve the equation

$$[J]\begin{Bmatrix} d\alpha_1 \\ d\beta_1 \\ d\alpha_2 \\ d\beta_2 \\ \vdots \\ d\alpha_k \\ d\beta_k \end{Bmatrix} = -\begin{Bmatrix} \Re_{11} \\ \Re_{12} \\ \vdots \\ \Re_{mn} \end{Bmatrix} \tag{15}$$

for $\{d\alpha_1 \ d\beta_1 \ d\alpha_2 \ d\beta_2 \ \ldots d\alpha_k \ d\beta_k\}^{\mathrm{T}}$,
(e) update the crack parameters

$$(\alpha_i)_{\text{new}} = (\alpha_i)_{\text{old}} + d\alpha_i, \quad (\beta_i)_{\text{new}} = (\beta_i)_{\text{old}} + d\beta_i \ (i = 1, 2, \ldots, k) \tag{16}$$

(f) iterate the procedures (b)–(e) until the residuals become sufficiently small.

It is important the number of equations mn exceeds the number of crack parameters $2k$ so that the system is sufficiently overdetermined, and Eq. (15) is solved in a least-squares sense using the singular value decomposition method. The elements of the Jacobian matrix are the sensitivities of the vibration amplitudes w_{ij} with respect to the crack parameters and they are computed numerically. For example, $\partial w_{ij}/\partial\alpha_k$ is computed by

$$\frac{\partial w_{ij}}{\partial \alpha_k} = \frac{w_{ij}(\alpha_1, \beta_1, \ldots, \alpha_k + \delta, \ldots, \beta_n) - w_{ij}(\alpha_1, \beta_1, \ldots, \alpha_k, \ldots, \beta_n)}{\delta}, \quad (|\delta| \ll 1) \tag{17}$$

The forward problem is solved $m(2k+1)$ times per iteration to build the Jacobian matrix and the residuals. To avoid overshoots in the early stage an underrelaxation is performed during the first several iterations

$$(\alpha_i)_{\text{new}} = (\alpha_i)_{\text{old}} + 0.25d\alpha_i, \quad (\beta_i)_{\text{new}} = (\beta_i)_{\text{old}} + 0.25d\beta_i \quad (i = 1, 2, \ldots, k) \tag{18}$$

The number of cracks present in a beam is usually unknown, and the identification of multiple cracks in a beam assuming that the number of cracks is known *a priori* has a serious limitation. To overcome this difficulty an additional procedure is proposed. When the number of cracks in a beam is unknown *a priori*, we assume that the beam has a single crack and solve the inverse problem to estimate the crack parameters and the square root of the residual sum of squares $(\sum_{i=1}^{m}\sum_{j=1}^{n}\Re_{ij}^2)^{1/2}$. We also assume several different numbers of cracks in the beam and solve the inverse problem to estimate the crack parameters and the square root of the residual sum of squares for each assumed crack number. Then we can find the most probable number of cracks by looking for the number which yields the minimum square root of the residual sum of squares.

To demonstrate the above procedure an example is provided. Let us suppose that the cracks in a beam are to be identified and that the number of cracks is unknown *a priori*. The deflections w_{ij}'s given in Table 2, which are the results of the harmonic response analyses by ANSYS, are input as the vibration amplitude measurements. The torsional stiffness of the rotational spring model is selected as $\kappa = 0.65\kappa_2$. Firstly the number of cracks in the beam is assumed to be one and the crack parameters and the square root of the residual sum of squares are computed. The number of cracks in the beam is assumed to be two and three, respectively, and the corresponding crack parameters and the square root of the residual sum of squares are computed, which are listed in Table 3. It shows that $(\sum_{i=1}^{m}\sum_{j=1}^{n}\Re_{ij}^2)^{1/2}$ is the smallest when the assumption of the number of cracks is made correctly, which indicates that the number of cracks in the beam is two. The estimated crack parameters are $\alpha_1 = 0.2252$, $\alpha_2 = 0.2609$, $\beta_1 = 0.3214$ and $\beta_2 = 0.6764$. The estimated crack locations are considered very accurate. Since the actual normalized crack locations are $\beta_1 = 0.3182$ and $\beta_2 = 0.6812$, the observed errors of the normalized crack locations are within ± 0.0048. On the other hand, the errors of the estimated crack sizes are within 13 percent of the actual sizes. It is supposed that the accuracy of the crack size estimations can be enhanced greatly by an improved torsional stiffness model.

Proper selection of the initial guesses of the crack parameters is important because the present method is based on the Newton–Raphson iteration method. The ranges of the initial guesses to produce a converged solution vary from case to case. When the number of cracks is assumed to be two in the crack identification given above, the solution does not

Table 3
Actual and detected crack parameters with the number of cracks unknown *a priori*.

	α_1	α_2	α_3	β_1	β_2	β_3	$(\sum_{i=1}^{m}\sum_{j=1}^{n}\Re_{ij}^2)^{1/2}$
Actual	0.2	0.3	–	0.3182	0.6812	–	
Detected (one crack assumed)	0.2977	–	–	0.3520	–	–	0.0971
Detected (two cracks assumed)	0.2252	0.2609	–	0.3214	0.6764	–	0.0067
Detected (three cracks assumed)	0.2112	0.2729	0.0118	0.3569	0.6870	0.8429	0.2777

Actual number of cracks is 2, $L = 0.8$ m, $b = 0.02$ m, $h = 0.02$ m, $m = 4$, $n = 3$, and $\kappa = 0.65\kappa_2$.

Table 4
Range of the initial guess of β_2 for $0.12 \leqslant \beta_1 \leqslant 0.47$ for the convergence of the solution.

β_1	β_2
$\beta_1 = 0.12$	$0.66 \leq \beta_2 \leq 0.85$
$\beta_1 = 0.2$	$0.56 \leq \beta_2 \leq 0.92$
$\beta_1 = 0.3$	$0.53 \leq \beta_2 \leq 0.91$
$\beta_1 = 0.4$	$0.51 \leq \beta_2 \leq 0.81$
$\beta_1 = 0.47$	$0.53 \leq \beta_2 \leq 0.74$

Two cracks assumed, actual crack parameters: $\alpha_1 = 0.2$, $\alpha_2 = 0.3$, $\beta_1 = 0.3182$, and $\beta_2 = 0.6812$.

Table 5
Mean values and standard deviations of the crack parameters when the vibration amplitude measurements are exposed to the random noise which is normally distributed with zero mean and standard deviation σ_{noise}.

σ_{noise}	$\overline{\alpha_1}$	σ_{α_1}	$\overline{\alpha_2}$	σ_{α_2}	$\overline{\beta_1}$	σ_{β_1}	$\overline{\beta_2}$	σ_{β_2}	$\overline{(\sum_{i=1}^{m}\sum_{j=1}^{n}\Re_{ij}^2)^{1/2}}$
0.001	0.2253	0.0010	0.2607	0.0013	0.3212	0.0013	0.6765	0.0011	0.0069
0.002	0.2255	0.0026	0.2605	0.0026	0.3215	0.0037	0.6764	0.0028	0.0086
0.005	0.2230	0.0082	0.2603	0.0053	0.3139	0.0115	0.6810	0.0080	0.0159
0.01	0.2262	0.0121	0.2641	0.0140	0.3162	0.0193	0.6774	0.0113	0.0256
0.02	0.2362	0.0224	0.2632	0.0255	0.3225	0.0260	0.6680	0.0232	0.0495

converge for the initial guess of $\beta_1 < 0.12$ or $\beta_1 > 0.47$. The range of initial guess of β_2 for the convergence of the solution is given in Table 4 for $0.12 \leq \beta_1 \leq 0.47$. The convergence of the solution is not strongly affected by the variations of the initial guesses of crack sizes, and initial values of $\alpha_1 = \alpha_2 = 0.3$ are used throughout this study.

Like all other inverse problems, the present method is influenced by input noise. To simulate the input noise, random numbers are generated so that they are normally distributed with zero mean and standard deviation σ_{noise}. The level of noise is controlled so that σ_{noise} becomes 0.001, 0.002, 0.005, 0.01 and 0.02, respectively. The random numbers are added to the amplitude measurements of Table 2 and the inverse problem is solved assuming that the number of cracks is two. The crack parameters are estimated twenty times for each noise level, and the standard deviations σ_{α_1}, σ_{α_2}, σ_{β_1} and σ_{β_2} as well as mean values $\overline{\alpha_1}$, $\overline{\alpha_2}$, $\overline{\beta_1}$, $\overline{\beta_2}$ and $\overline{(\sum_{i=1}^{m}\sum_{j=1}^{n}\Re_{ij}^2)^{1/2}}$ for each noise level are calculated as given in Table 5. It shows that $\overline{\alpha_1}$, $\overline{\alpha_2}$, $\overline{\beta_1}$ and $\overline{\beta_2}$ remain much the same even though σ_{α_1}, σ_{α_2}, $\sigma_{\beta_1} \sigma_{\beta_2}$ and $\overline{(\sum_{i=1}^{m}\sum_{j=1}^{n}\Re_{ij}^2)^{1/2}}$ increase as σ_{noise} increases. It is strongly recommended to estimate the crack parameters by averaging the solutions of the inverse problems many times because the input noise is always present in the measurements.

4. Conclusions

A simple method to detect multiple cracks in a beam using the vibration amplitudes is presented. The cracks are modeled as massless rotational springs and the forward problem is solved by using the finite element method based on the Euler–Bernoulli beam theory. The inverse problem is solved iteratively for the crack locations and sizes by the Newton–Raphson method. The number of equations exceeds the number of crack parameters so that the system is overdetermined, and the system of equations is solved in a least-squares sense using the singular value decomposition method. A two-dimensional finite element model using ANSYS is built to simulate the experimental tests of Ruotolo and Surace [21], and vibration amplitude measurements based on the two-dimensional finite element model are obtained. It is found that there exists a significant discrepancy between the torsional stiffness models of Ostachowicz and Krawkczuk [18] and Dimarogonas and Paipetis [20] and that their application without proper calibration may lead to an erroneous solution.

Scaled down torsional stiffness model of Dimarogonas and Paipetis [20] is used in this study. The difficulty of identifying multiple cracks without *a priori* knowledge of the number of cracks is overcome by comparing the residual sum of squares of each solution with assumed number of cracks. The observed errors of the normalized crack locations β_1 and β_2 are within ± 0.0048, and the errors of the estimated crack sizes are within 13 percent of the actual sizes. It is supposed that the accuracy of the crack size estimations can be enhanced greatly by an improved torsional stiffness model. Choice of the initial guesses of the crack parameters is important because the present method is based on the Newton–Raphson iteration method. The ranges of the initial guesses to produce a converged solution are investigated. Also it is shown how the solution of inverse problem is affected by the input data noise.

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